# 57

02 November 2019 19:05

$|S| = n$. $\mathcal{F} = \{f_i : S \to S\}$. $|\mathcal{F}| = n^n$

$p_i = \Pr[\text{choosing } f_i]$ ; $\sum_i p_i = 1$.

$\tilde{P}_{ss'} = \sum_{f_i : f(s) = s'} p_i$ ; $\sum_{s'} P_{ss'} = 1$, since

$\mathcal{F}_s = \bigcup_{s'} \{f_i : f_i(s) = s'\} = \mathcal{F}$.

Let $P$ be the input matrix.
Define $Q = P$, start with $\tilde{P} = 0$.

$Q_i = \{q_{ss'} : f_i(s) = s'\}$ ; $q_i = \min Q_i$

```
for f_i in F:
    if q_i = 0:
        continue.
    else:
        p_i := q_i
```

for $q_{ss'}$ in $Q_i$:

$q_{ss'} = q_{ss'} - q_i$

recompute $q_i$'s

return $\tilde{P}$.

Define $\tilde{P}_i = \{ \tilde{P}_{ss'} : f_i(s) = s' \}$.

Claim: $Q + \tilde{P}$ is invariant at each iteration.

At each iteration, either nothing happens, or $q_{ss'} = q_{ss'} - q_i$, $\tilde{P}_{ss'} = \tilde{P}_{ss'} + q_i$.

Claim: At end of iteration $i$, $q_i = 0$.

Either in the beginning, $q_i = 0$, or $q_i > 0$, $q_i = \min Q_i$ at some $q_{tt'}$.

At the end of the iteration, $q_{tt'} = q_{tt'} - q_i$

$= 0$

$\therefore q_i = \min Q_i = 0$.

Claim: Row sum of $\tilde{P}$ is invariant.

$\forall s, \sum_{s'} p_{ss'} = \sum_{i \in \mathcal{F}_s} f_i = \sum_{i \in \mathcal{F}_1} f_i$

This also implies row sum of Q is inv.

Claim: $Q = 0$ at the last iteration.

Suppose $q_{ss'} > 0$. Then every row has a $q_{s_i s_i'} > 0$. Consider $f_\alpha : s_i \mapsto s_i'$. $q_\alpha > 0$. at final iteration. Contradiction.

Initially $Q = P$, $\tilde{P} = 0$. Finally $Q = 0$.

$$\therefore \tilde{P} = P.$$